A Novel Class of Intermediate Shapes Between Squares and Circles

Mathematics Extended Essay

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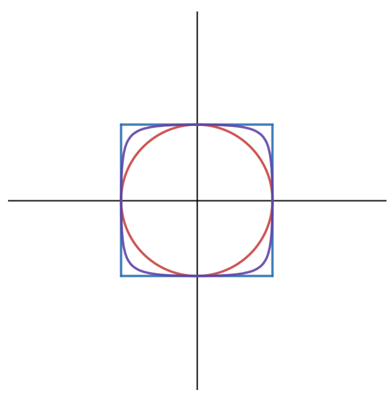

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1 Introduction

In two-dimensional Euclidean geometry, distance is given by the equation

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

which can be rewritten in the form

$$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

According to [1], "The word *distance* can take on different meanings depending upon what particular space one is talking about." The example given is that the distance between two points on a globe is not found using the standard Euclidean formula. From this premise, the idea crossed my mind to explore a non-Euclidean geometry [2] in which distance is defined differently. My first idea for an alternate distance rule was to reverse the order of the exponents, leading to the form

$$2^d = 2^{(x_2 - x_1)} + 2^{(y_2 - y_1)}$$

However, two issues immediately arose with that definition: distances in the positive and negative directions were different, and the distance between points that shared a coordinate was not equal to the positive difference of the other coordinates. These were solved by modifying the formula to

$$2^{d^2} = 2^{(x_2 - x_1)^2} + 2^{(y_2 - y_1)^2} - 1$$

Unfortunately, that concept of distance created unresolvable paradoxes. For example, there was no consistent way to define the shortest path between two points. Despite the inconsistency, I continued examining the equation and noticed that the sets of points of equal "distance" from the origin formed interesting shapes.

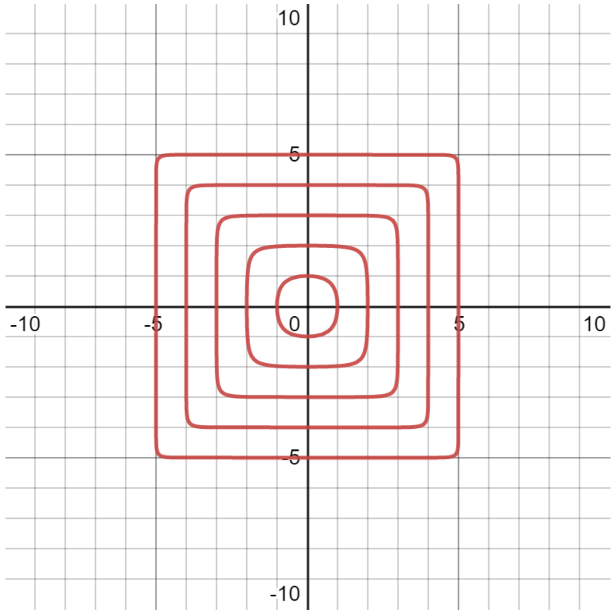

Figure 1: Points with "distances" of 1, 2, 3, 4, and 5 from the origin [3]

As seen in figure 1, the curves become more circular as they approach the origin and become more square as they get farther away. Scaling the curves down by an amount proportional to their "distance" from the origin, I was able to create a general formula that could produce shapes between squares and circles given a variable *s* called the squareness factor. This led directly to the discovery of logarithmic squircles and the formulation of my research question: "What are the properties of logarithmic squircles?"

2 Logarithmic Squircles

A logarithmic squircle is a shape consisting of a set of points that satisfies

$$e^{s^2} = e^{\frac{s^2}{r^2}x^2} + e^{\frac{s^2}{r^2}y^2} - 1$$

or any shape congruent to such a set.

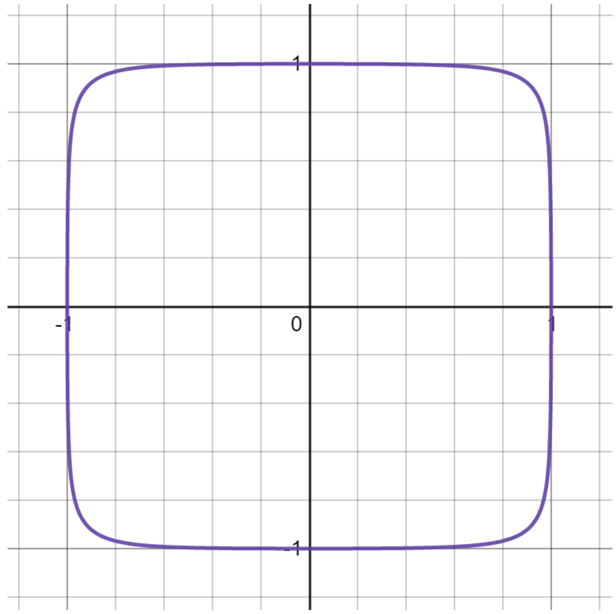

Figure 2: A logarithmic squircle with $r = 1$ and $s = 2$ [3]

In the equation, s is the squareness factor, and r is the radius. Note that in this context "radius" is defined as half of the shortest distance across the shape that passes through the origin. The squareness factor and the radius must both fall somewhere on the interval $(0, \infty)$.

3 General Squircles

3.1 Common Squircles

According to [4], "There are two incompatible definitions of the squircle." The first is a special case of the superellipse:

$$x^4 + y^4 = a^4$$

where a is a constant. The general superellipse was discovered by Lamé [5] and can be expressed as

$$|\frac{x}{a}|^r + |\frac{y}{b}|^r = 1$$

This equation produces a circle when $r = 2$ and $a = b$, and it approaches a square as r approaches ∞ with $a = b$. Consequently, all superellipses with $r \geq 2$ and $a = b$ can be thought of as squircles with varying levels of squareness. These will be called Lamé squircles. The other definition of a squircle is the Fernandez-Guasti squircle, which is given by

$$s^2\frac{x^2}{k^2}\frac{y^2}{k^2} - (\frac{x^2}{k^2} + \frac{y^2}{k^2}) + 1 = 0$$

In this case, s acts as the squareness factor, with $s = 0$ giving a circle and $s = 1$ giving a square. As will be mentioned later, these are not the only types of squircles, but they are the most common.

3.2 Broad Definition

The two common types of squircles are defined completely independently of each other, so there is not a single list of criteria that makes a squircle. In order to show that logarithmic squircles are squircles, such a list must be created. The following properties are proposed to solve this problem and are based on the intuitive notion of an intermediate shape:

1. A squircle is a closed shape that can lie between a square and the largest circle that can be inscribed within the square.
2. A squircle touches the square and the circle, when they are drawn, at the points where they intersect each other but touches neither at any other location. These four points are herein referred to as n-points because the space between the square and circle narrows there.
3. A squircle is a convex shape that has no straight edges.
4. A squircle is symmetric along all of the axes along which its bounding square is symmetric.

The first property is simply a more rigorous way of defining what it means for a squircle to be between a square and a circle. The second property clarifies the first by stating that there should be four points of contact between the squircle and its bounding shapes. This is necessary since the squircle could not be closed if it was not allowed to touch the bounding shapes at the n-points. The third property is required to distinguish squares with rounded edges from squircles and to prevent squircles from being dented. Finally, the fourth property is needed to ensure that a squircle resembles its bounding square.

4 Proofs of Properties

In this section, it will be proven that logarithmic squircles uphold the four properties of a squircle. To assist in these proofs, several observations will be made. First note that the r^2 term in the equation for a logarithmic squircle dilates the curve along the *x* and *y* axes equally, so changing *r* and letting *s* remain the same will produce classes of similar shapes. As such, proving the properties of a squircle apply in the case $r = 1$ will simultaneously prove the same for all $r \in (0, \infty)$.

$$e^{s^2} = e^{\frac{s^2}{r^2}x^2} + e^{\frac{s^2}{r^2}y^2} - 1$$

will be simplified to

$$e^{s^2} = e^{s^2x^2} + e^{s^2y^2} - 1$$

and $r = 1$ will be assumed. Also note that the previous equation can be algebraically rearranged:

$$e^{s^2y^2} = e^{s^2} - e^{s^2x^2} + 1$$

$$s^2y^2 = \ln(e^{s^2} - e^{s^2x^2} + 1)$$

$$sy = \pm\sqrt{\ln(e^{s^2} - e^{s^2x^2} + 1)}$$

$$y = \frac{\pm\sqrt{\ln(e^{s^2} - e^{s^2x^2} + 1)}}{s}$$

Omitting the negative part allows for the creation of a two-variable function:

$$q(x,s) = \frac{\sqrt{\ln(e^{s^2} - e^{s^2x^2} + 1)}}{s}$$

Finally, notice from the function that if x^2 were greater than 1, the argument to the logarithm would be less than 1, which would either make the logarithm's output negative or undefined. Neither case provides a real output to the function, so the domain of $q(x,s)$ is $x \in [-1,1]$.

4.1 First Property

The proof of the first property will be broken down into several parts. First, it will be shown that logarithmic squircles are closed curves. Afterwards, it will be demonstrated that logarithmic squircles lie between a square and the largest circle that can be inscribed within the square through three criteria:

1. As s approaches 0, $q(x,s)$ will approach a semicircle with radius 1.
2. As s approaches ∞, $q(x,s)$ will approach the top of a square with a side length of 2.
3. As s increases and x remains the same, $q(x,s)$ will not decrease anywhere on $[-1,1]$.

These criteria will be sufficient for a proof because the third criterion ensures that no part of the curve will go above the upper limit or below the lower. It should also be noted that a circle of radius 1 is the largest circle that can be inscribed in a square with side length 2, so that aspect of the property will be satisfied.

4.1.1 Closed Curve

To prove that logarithmic squircles are closed shapes, it must be shown that their positive and negative sections are both continuous and that they are connected to each other. The sections are connected at $(-1,0)$ and $(1,0)$ because $q(\pm 1,s) = -q(\pm 1,s) = 0$ for all $s \in (0,\infty)$. Proving continuity will first require the range of $e^{s^2} - e^{s^2x^2} + 1$ over the domain $x \in [-1,1]$ to be found.

To find the maximum value of the expression, the negative part will be minimized, which can be done by substituting in 0 for *x*:

$$e^{s^2} - e^{s^2(0)^2} + 1 = e^{s^2} - 1 + 1 = e^{s^2}$$

The minimum can be found by maximizing the negative part, which is done by either substituting -1 or 1 for *x*:

$$e^{s^2} - e^{s^2(1)^2} + 1 = e^{s^2} - e^{s^2} + 1 = 1$$

Hence, the range is $e^{s^2} - e^{s^2x^2} + 1 \in [1, e^{s^2}]$. When treating *s* as a constant in $(0, \infty)$, the expression is of the form $a - b^{x^2}$ such that *a* and *b* are positive real numbers, so continuous values are produced for all real *x*. The expression produces continuous values in the range $[1, e^{s^2}]$, the numbers in that set are all positive reals, and the natural logarithm is continuous for all positive real numbers, so the natural logarithm of the expression will also produce continuous values. Taking this logarithm makes a new expression: $\ln(e^{s^2} - e^{s^2x^2} + 1)$. The maximum of this expression is found by taking the natural log of the maximum value produced by the previous expression:

$$\ln(e^{s^2}) = s^2$$

The minimum is found by doing the same to the previous minimum:

$$\ln(1) = 0$$

Hence, it outputs values in the range $[0, s^2]$. Finally, the square root of this new expression can be taken and divided by *s* to produce something equal to $q(x, s)$. The expression that had its square root taken produces continuous values in the range $[0, s^2]$, the numbers in that set are all non-negative reals, and the square root function divided by a constant is continuous for all non-negative real numbers, so $q(x, s)$ is continuous for all $x \in [-1, 1]$. Further note that $-q(x, s)$ is just a reflection of the prior across the *x* axis, so it is also continuous. Therefore, logarithmic squircles are closed curves.

4.1.2 Criterion I

The first criterion is proven by using L'Hopital's rule [6]:

$$f(x) = \lim_{s\to 0} q(x,s) = \lim_{s\to 0} \frac{\sqrt{\ln(e^{s^2} - e^{s^2x^2} + 1)}}{s}$$

$$f^2(x) = \lim_{s\to 0} \frac{\ln(e^{s^2} - e^{s^2x^2} + 1)}{s^2}$$

$$\sqrt{\ln(e^{(0)^2} - e^{(0)^2x^2} + 1)} = \sqrt{\ln(1 - 1 + 1)} = 0$$

$$(0)^2 = 0$$

$$\therefore f^2(x) = \lim_{s\to 0} \frac{\frac{\partial}{\partial s}\ln(e^{s^2} - e^{s^2x^2} + 1)}{\frac{\partial}{\partial s}s^2}$$

$$= \lim_{s\to 0} \frac{\frac{2se^{s^2} - 2x^2se^{s^2x^2}}{e^{s^2} - e^{s^2x^2} + 1}}{2s} = \lim_{s\to 0} \frac{e^{s^2} - x^2e^{s^2x^2}}{e^{s^2} - e^{s^2x^2} + 1} = \frac{e^{(0)^2} - x^2e^{(0)^2x^2}}{e^{(0)^2} - e^{(0)^2x^2} + 1} = \frac{1 - x^2}{1}$$

$$\therefore f(x) = \sqrt{1 - x^2}$$

$$\lim_{s\to 0} q(x,s) = \sqrt{1 - x^2}$$

Because $y = \sqrt{1 - x^2}$ is the equation of a semicircle with radius 1, the first criterion is true.

4.1.3 Criterion II

To prove the second criterion, it will be shown that $q(x,s)$ lies between two functions that approach 1 as s approaches ∞, thus showing that $q(x,s)$ approaches 1 by a squeeze argument. For the lower bound, a new function is defined as follows:

$$\text{ק}(x,s) = \frac{\sqrt{\ln(e^{s^2} - e^{s^2x^2})}}{s} : s \in (0, \infty)$$

The domain of this function should be found. For it to be real, the value inside the natural logarithm must be positive, and the value under the square root must be non-negative. Starting with the part within the logarithm,

$$e^{s^2} - e^{s^2x^2} > 0$$

$$e^{s^2} > e^{s^2x^2}$$

$$s^2 > s^2x^2$$

$$x^2 < 1$$

This shows that the domain is at most $x \in (-1, 1)$. For further specification, the part under the square root is examined:

$$\ln(e^{s^2} - e^{s^2x^2}) \geq 0$$

$$e^{s^2} - e^{s^2x^2} \geq 1$$

$$e^{s^2} - 1 \geq e^{s^2x^2}$$

$$\ln(e^{s^2} - 1) \geq s^2x^2$$

$$x^2 \leq \frac{\ln(e^{s^2} - 1)}{s^2}$$

Because $\ln(e^{s^2} - 1)$ will always be less than s^2 for the allowed values of *s*, this creates the more restrictive domain: $x \in [-\frac{\sqrt{\ln(e^{s^2}-1)}}{s}, \frac{\sqrt{\ln(e^{s^2}-1)}}{s}]$. Next, $ק(x, s)$ can be manipulated as so:

$$\frac{\sqrt{\ln(e^{s^2} - e^{s^2x^2})}}{s} = \sqrt{\frac{\ln(e^{s^2} - e^{s^2x^2})}{s^2}} = \sqrt{\frac{\ln(e^{s^2} - e^{s^2x^2})}{\ln(e^{s^2})}} = \sqrt{\log_{e^{s^2}}(e^{s^2} - e^{s^2x^2})}$$

$$= \sqrt{\log_{e^{s^2}}(e^{s^2}(1 - e^{s^2(x^2-1)}))} = \sqrt{1 + \log_{e^{s^2}}(1 - e^{s^2(x^2-1)})}$$

Based on this rearranged version of the function, the limit as *s* approaches ∞ can be found:

$$\lim_{s\to\infty} ק(x, s) = \lim_{s\to\infty} \sqrt{1 + \log_{e^{s^2}}(1 - e^{s^2(x^2-1)})}$$

Since $x \in (-1, 1)$ for all x in the domain of the function, x^2-1 is negative. Therefore, $s^2(x^2-1)$ approaches $-\infty$, $e^{s^2(x^2-1)}$ approaches 0, and $1 - e^{s^2(x^2-1)}$ approaches 1. Additionally, e^{s^2} approaches ∞. This means that $\lim_{s\to\infty} \log_{e^{s^2}}(1 - e^{s^2(x^2-1)})$ fits both of the following forms: $\lim_{a\to 1} \log_b(a)$ and $\lim_{b\to\infty} \log_b(a)$. Both of these tend to 0, so the limit is

$$\lim_{s\to\infty} \text{ק}(x, s) = \sqrt{1 + 0} = 1$$

Now realize that the square root and natural logarithm functions are both strictly increasing over their domains and that s is always positive, so $\frac{\sqrt{\ln(x)}}{s}$ will be strictly increasing as x increases, given that the expression is defined. Additionally realize that $e^{s^2} - e^{s^2x^s} + 1 > e^{s^2} - e^{s^2x^2}$. Therefore,

$$\frac{\sqrt{\ln(e^{s^2} - e^{s^2x^2} + 1)}}{s} > \frac{\sqrt{\ln(e^{s^2} - e^{s^2x^2})}}{s}$$

$$q(x, s) > \text{ק}(x, s)$$

given that both functions are defined for the specified x and s values. For an upper bound, the function $f(x) = 1$ will be used. Given that $x \in [-1, 1]$, it is impossible for $q(x, s)$ to be greater than 1 because its maximum value is found by minimizing $e^{s^2x^2}$, which happens when $x = 0$:

$$\frac{\sqrt{\ln(e^{s^2} - e^{s^2(0)^2} + 1)}}{s} = \frac{\sqrt{\ln(e^{s^2} - 1 + 1)}}{s} = \frac{\sqrt{\ln(e^{s^2})}}{s} = 1$$

Hence, $\text{ק}(x, s) < q(x, s) \leq 1$ given that both expressions are defined for the specified x and s values. Although $\text{ק}(x, s)$ is not always defined over the interval $x \in (-1, 1)$, it will be for sufficiently large s. Recall that $\lim_{s\to\infty} \text{ק}(x, s)$ was found for all $x \in (-1, 1)$. This means that, once s is large enough, $\text{ק}(x, s)$ will simplify to the same case shown in the limit as long as $x \in (-1, 1)$. Therefore, the inequality will eventually hold for all x in that interval. As such, $\lim_{s\to\infty} q(x, s)$ is between two values that both approach 1. By the squeeze theorem, $\lim_{s\to\infty} q(x, s) = 1 : x \in (-1, 1)$. It has already been shown that $q(x, s) = 0$ for $x = -1$ or $x = 1$, so the middle of the function will approach the line $y = 1$, and the two extreme values will remain 0. The function is continuous, so the regions around the extreme values must approach $x = -1$ and $x = 1$, respectively, such that $q(x, s) \in [0, 1)$. The shape described is the top half of a square with side length 2.

4.1.4 Criterion III

The only remaining criterion is the third, which can be proven by showing that the derivative of $q(x, s)$ with respect to s is non-negative. To start, the quotient rule is used to take the derivative:

$$\frac{\partial}{\partial s} q(x, s) = \frac{\partial}{\partial s} \frac{\sqrt{\ln(e^{s^2} - e^{s^2 x^2} z + 1)}}{s}$$

$$= \frac{s \frac{\partial}{\partial s} \sqrt{\ln(e^{s^2} - e^{s^2 x^2} + 1)} - \sqrt{\ln(e^{s^2} - e^{s^2 x^2} + 1)}}{s^2}$$

$$= \frac{s \frac{\frac{2se^{s^2} - 2x^2 s e^{s^2 x^2}}{e^{s^2} - e^{s^2 x^2} + 1}}{2\sqrt{\ln(e^{s^2} - e^{s^2 x^2} + 1)}} - \sqrt{\ln(e^{s^2} - e^{s^2 x^2} + 1)}}{s^2}$$

$$= \frac{e^{s^2} - x^2 e^{s^2 x^2}}{(e^{s^2} - e^{s^2 x^2} + 1)\sqrt{\ln(e^{s^2} - e^{s^2 x^2} + 1)}} - \frac{\sqrt{\ln(e^{s^2} - e^{s^2 x^2} + 1)}}{s^2}$$

The end result is not obviously non-negative. To show that it is, it must be manipulated further:

$$\frac{\partial}{\partial s} q(x, s) = \frac{s^2(e^{s^2} - x^2 e^{s^2 x^2}) - (e^{s^2} - e^{s^2 x^2} + 1) \ln(e^{s^2} - e^{s^2 x^2} + 1)}{s^2(e^{s^2} - e^{s^2 x^2} + 1)\sqrt{\ln(e^{s^2} - e^{s^2 x^2} + 1)}}$$

Each of the three terms in the denominator is non-negative, so the denominator as a whole is as well. All that remains to be shown is that the numerator is non-negative. This will be proven by showing that

$$s^2(e^{s^2} - x^2 e^{s^2 x^2}) = (e^{s^2} - e^{s^2 x^2} + 1) \ln(e^{s^2} - e^{s^2 x^2} + 1)$$

when $s = 0$ and that

$$\frac{\partial}{\partial s} s^2(e^{s^2} - x^2 e^{s^2 x^2}) \geq \frac{\partial}{\partial s} (e^{s^2} - e^{s^2 x^2} + 1) \ln(e^{s^2} - e^{s^2 x^2} + 1)$$

for all $s \in (0, \infty)$ and $x \in [-1, 1]$. This method works because if two functions are equal initially but one increases at a rate greater than or equal to that of the other, the one with the lesser rate of change can never grow fast enough to surpass the one with the greater rate of change. The first part is shown using substitution:

$$(0)^2(e^{(0)^2} - x^2 e^{(0)^2 x^2}) = (e^{(0)^2} - e^{(0)^2 x^2} + 1) \ln(e^{(0)^2} - e^{(0)^2 x^2} + 1)$$

$$0(1 - x^2(1)) = (1 - 1 + 1) \ln(1 - 1 + 1)$$

$$0(1 - x^2) = 1 \ln(1)$$

$$0 = 0$$

The second part is shown by first taking the derivative of each:

$$\frac{\partial}{\partial s} s^2(e^{s^2} - x^2 e^{s^2 x^2}) = 2s(e^{s^2} - x^2 e^{s^2 x^2}) + s^2(2se^{s^2} - 2x^4 se^{s^2 x^2})$$

$$\frac{\partial}{\partial s}(e^{s^2} - e^{s^2 x^2} + 1) \ln(e^{s^2} - e^{s^2 x^2} + 1) =$$

$$(2se^{s^2} - 2x^2 se^{s^2 x^2}) \ln(e^{s^2} - e^{s^2 x^2} + 1) + (2se^{s^2} - 2x^2 se^{s^2 x^2})$$

The inequality holds if the first of the two derivatives minus the second is non-negative, so they are subtracted to test this:

$$[2s(e^{s^2} - x^2 e^{s^2 x^2}) + s^2(2se^{s^2} - 2x^4 se^{s^2 x^2})] -$$

$$[(2se^{s^2} - 2x^2 se^{s^2 x^2}) \ln(e^{s^2} - e^{s^2 x^2} + 1) + (2se^{s^2} - 2x^2 se^{s^2 x^2})] =$$

$$[2s(e^{s^2} - x^2 e^{s^2 x^2}) - (2se^{s^2} - 2x^2 se^{s^2 x^2})] +$$

$$[s^2(2se^{s^2} - 2x^4 se^{s^2 x^2}) - (2se^{s^2} - 2x^2 se^{s^2 x^2}) \ln(e^{s^2} - e^{s^2 x^2} + 1)] =$$

$$s^2(2se^{s^2} - 2x^4se^{s^2x^2}) - (2se^{s^2} - 2x^2se^{s^2x^2})\ln(e^{s^2} - e^{s^2x^2} + 1)$$

To find the maximum value of $\ln(e^{s^2} - e^{s^2x^2} + 1)$, $e^{s^2x^2}$ must be minimized by substituting 0 for x:

$$\ln(e^{s^2} - e^{s^2(0)^2} + 1) = \ln(e^{s^2} - 1 + 1) = s^2$$

Based on this maximum, $\ln(e^{s^2} - e^{s^2x^2} + 1) \leq s^2$, with the two only being equal at $x = 0$. It is also apparent that $2se^{s^2} - 2x^2se^{s^2x^2} \leq 2se^{s^2} - 2x^4se^{s^2x^2}$, again with the two only equal at $x = 0$, because $x^2 \geq x^4$ for all $x \in [-1, 1]$, thereby making the negative part of the left hand side of the inequality greater than or equal to that of the right. Given that the positive parts are equal, this makes the left side less than or equal to the right. These two inequalities together show the following:

$$s^2(2se^{s^2} - 2x^4se^{s^2x^2}) = ab$$

$$(2se^{s^2} - 2x^2se^{s^2x^2})\ln(e^{s^2} - e^{s^2x^2} + 1) = cd$$

$$a \geq c$$

$$b \geq d$$

$$\therefore s^2(2se^{s^2} - 2x^4se^{s^2x^2}) \geq (2se^{s^2} - 2x^2se^{s^2x^2})\ln(e^{s^2} - e^{s^2x^2} + 1)$$

Hence, the difference of the derivatives is not negative, so

$$\frac{\partial}{\partial s}s^2(e^{s^2} - x^2e^{s^2x^2}) \geq \frac{\partial}{\partial s}(e^{s^2} - e^{s^2x^2} + 1)\ln(e^{s^2} - e^{s^2x^2} + 1)$$

This means that the derivative of $q(x, s)$ with respect to s is non-negative, so the third criterion is true. Since it has been proven that logarithmic squircles are closed curves and that the first, second, and third criteria are true, thereby proving that logarithmic squircles lie between a circle with radius 1 and a square with side length 2, the first property is upheld.

4.2 Second Property

Using the equation $x^2 + y^2 = 1$ for a circle with radius 1 and the equations $y = -1$, $y = 1$, $x = -1$, and $x = 1$ to trace the outside of a square with side length 2, the location of the n-points can be found. This is done by substituting the given x or y value for each side of the square into the equation of the circle and solving the equation for the other value:

$$x^2 + (-1)^2 = 1$$

$$x^2 + 1 = 1$$

$$x = 0$$

gives the point $(0, -1)$.

$$x^2 + (1)^2 = 1$$

$$x^2 + 1 = 1$$

$$x = 0$$

gives the point $(0, 1)$.

$$(-1)^2 + y^2 = 1$$

$$1 + y^2 = 1$$

$$y = 0$$

gives the point $(-1, 0)$. Finally,

$$(1)^2 + y^2 = 1$$

$$1 + y^2 = 1$$

$$y = 0$$

gives the point $(1, 0)$. To verify that these are the only places that the logarithmic squircle touches the square, the sides of the square are substituted into the logarithmic squircle's equation:

For $y = -1$:

$$e^{s^2} = e^{s^2x^2} + e^{s^2(-1)^2} - 1$$

$$1 = e^{s^2x^2}$$

$$x = 0$$

For $y = 1$:

$$e^{s^2} = e^{s^2x^2} + e^{s^2(1)^2} - 1$$

$$1 = e^{s^2x^2}$$

$$x = 0$$

For $x = -1$:

$$e^{s^2} = e^{s^2(-1)^2} + e^{s^2y^2} - 1$$

$$1 = e^{s^2y^2}$$

$$y = 0$$

For $x = 1$:

$$e^{s^2} = e^{s^2(1)^2} + e^{s^2y^2} - 1$$

$$1 = e^{s^2y^2}$$

$$y = 0$$

As shown above, the logarithmic squircle only touches the sides of the square at the n-points. Next, the intersections of the logarithmic squircle and the circle are found. To begin, the equation of the circle is rearranged:

$$x^2 + y^2 = 1$$

$$y^2 = 1 - x^2$$

Hence, y^2 is replaced with $1 - x^2$ by substitution:

$$e^{s^2} = e^{s^2x^2} + e^{s^2(1-x^2)} - 1$$

$$e^{s^2} = e^{s^2x^2} + e^{s^2-s^2x^2} - 1$$

$$e^{s^2} = e^{s^2x^2} + \frac{e^{s^2}}{e^{s^2x^2}} - 1$$

$$e^{s^2}e^{s^2x^2} = e^{2s^2x^2} + e^{s^2} - e^{s^2x^2}$$

$$(e^{s^2x^2})^2 - (e^{s^2} + 1)e^{s^2x^2} + e^{s^2} = 0$$

From here, the quadratic equation is used to solve for $e^{s^2x^2}$:

$$e^{s^2x^2} = \frac{e^{s^2} + 1 \pm \sqrt{(e^{s^2}+1)^2 - 4e^{s^2}}}{2} = \frac{e^{s^2} + 1 \pm \sqrt{(e^{s^2})^2 - 2e^{s^2} + 1}}{2} =$$

$$\frac{e^{s^2} + 1 \pm \sqrt{(e^{s^2}-1)^2}}{2} = \frac{e^{s^2} + 1 \pm (e^{s^2}-1)}{2}$$

Therefore $e^{s^2x^2}$ either equals e^{s^2} or 1. This implies that *x* equals -1, 1, or 0. Finally, the corresponding y values can be found for each of these by taking $\pm q(x, s)$ for all $x \in \{-1, 1, 0\}$. The value of *s* does not end up mattering. The points found are as follows:

$$(-1, \pm q(-1, s)) : (-1, 0)$$

$$(1, \pm q(1, s)) : (1, 0)$$

$$(0, \pm q(0, s)) : (0, -1), (0, 1)$$

These points are the n-points found earlier. Thus, the logarithmic squircle only touches the square and the circle at the four points where they touch each other, so the second property is upheld.

4.3 Third Property

The proof that logarithmic squircles are convex relies on the fact that closed shapes are convex if and only if they are concave down on the top and concave up on the bottom, with points with undefined second derivatives connecting the sections. A function is concave up when its second derivative is positive and concave down when it is negative [7]. To show that this is true of a logarithmic squircle, the second derivative must be taken with respect to x. It is simplest to do this implicitly:

$$\frac{\partial}{\partial x}e^{s^2} = \frac{\partial}{\partial x}(e^{s^2x^2} + e^{s^2y^2} - 1)$$

$$0 = 2s^2xe^{s^2x^2} + 2s^2ye^{s^2y^2}\frac{\partial y}{\partial x}$$

$$\frac{\partial y}{\partial x} = -\frac{xe^{s^2x^2}}{ye^{s^2y^2}}$$

$$\frac{\partial^2 y}{\partial x^2} = -\frac{ye^{s^2y^2}\frac{\partial}{\partial x}(xe^{s^2x^2}) - xe^{s^2x^2}\frac{\partial}{\partial x}(ye^{s^2y^2})}{y^2e^{2s^2y^2}} =$$

$$-\frac{ye^{s^2y^2}(e^{s^2x^2} + 2s^2x^2e^{s^2x^2}) - xe^{s^2x^2}(e^{s^2y^2} + 2y^2e^{s^2y^2})\frac{\partial y}{\partial x}}{y^2e^{2s^2y^2}} =$$

$$-\frac{ye^{s^2y^2}(e^{s^2x^2} + 2s^2x^2e^{s^2x^2}) + \frac{(xe^{s^2x^2})^2}{ye^{s^2y^2}}(e^{s^2y^2} + 2y^2e^{s^2y^2})}{y^2e^{2s^2y^2}} =$$

$$-\frac{y^2e^{2s^2y^2}(e^{s^2x^2} + 2s^2x^2e^{s^2x^2}) + x^2e^{2s^2x^2}(e^{s^2y^2} + 2y^2e^{s^2y^2})}{y^3e^{3s^2y^2}}$$

Because both terms in the numerator are the products of positive values, given neither x nor y is 0, their sum is positive. Therefore, the whole numerator is positive. The denominator is the product of y^3 and a value that will always be positive, so its sign depends exclusively on that of y^3. The whole second derivative is negated, and therefore its overall sign will always be the opposite of the sign of y^3. In the case $x = 0$, the numerator will still be positive, so the sign is still the opposite of that of y^3. In the case $y = 0$, the second derivative is undefined, representing the two boundary points between the top and bottom of the curve. Because of this correlation, the second derivative will be negative on the top half of a logarithmic squircle and positive on the bottom half. Thus, it will be concave down on the top and concave up on the bottom. Hence, logarithmic squircles are convex. It is also clear that there are no straight edges on a logarithmic squircle because straight lines have a second derivative of 0:

$$y = mx + b$$

$$\frac{dy}{dx} = m$$

$$\frac{d^2y}{dx^2} = 0$$

However, a logarithmic squircle's second derivative can never be 0:

$$-\frac{y^2e^{2s^2y^2}(e^{s^2x^2} + 2s^2x^2e^{s^2x^2}) + x^2e^{2s^2x^2}(e^{s^2y^2} + 2y^2e^{s^2y^2})}{y^3e^{3s^2y^2}} = 0$$

$$y^2e^{2s^2y^2}(e^{s^2x^2} + 2s^2x^2e^{s^2x^2}) + x^2e^{2s^2x^2}(e^{s^2y^2} + 2y^2e^{s^2y^2}) = 0$$

For the above equation to be true, x and y would both have to equal 0, but that is impossible since $(0, 0)$ does not lie on a logarithmic squircle. Thus, logarithmic squircles are convex shapes with no straight edges, and the third property is upheld.

4.4 Fourth Property

The bounding square for a logarithmic squircle has the following axes of symmetry: $y = 0$, $x = 0$, $y = x$, and $y = -x$. To demonstrate that logarithmic squircles are symmetric along these axes, a reflection will be performed on the equation along each axis. If a reflection returns the original equation, then logarithmic squircles are symmetric along that axis. To reflect along $y = 0$, $-y$ will be substituted for y:

$$e^{s^2} = e^{s^2x^2} + e^{s^2(-y)^2} - 1$$

$$e^{s^2} = e^{s^2x^2} + e^{s^2y^2} - 1$$

To reflect along $x = 0$, $-x$ will be substituted for x:

$$e^{s^2} = e^{s^2(-x)^2} + e^{s^2y^2} - 1$$

$$e^{s^2} = e^{s^2x^2} + e^{s^2y^2} - 1$$

To reflect along $y = x$, x will be substituted for y and vice versa:

$$e^{s^2} = e^{s^2(y)^2} + e^{s^2(x)^2} - 1$$

$$e^{s^2} = e^{s^2y^2} + e^{s^2x^2} - 1$$

$$e^{s^2} = e^{s^2x^2} + e^{s^2y^2} - 1$$

Finally, to reflect along $y = -x$, $-x$ will be substituted for y and $-y$ for x:

$$e^{s^2} = e^{s^2(-y)^2} + e^{s^2(-x)^2} - 1$$

$$e^{s^2} = e^{s^2y^2} + e^{s^2x^2} - 1$$

$$e^{s^2} = e^{s^2x^2} + e^{s^2y^2} - 1$$

Thus, logarithmic squircles are symmetric along all the axes that their bounding squares are, so the fourth property is upheld. Additionally, the proofs of the first, second, third, and fourth properties together demonstrate that logarithmic squircles are, in fact, squircles.

5 Non-equivalence

One more property that ought to be shown is that logarithmic squircles are not equivalent to other types of squircles. This idea can be precisely formulated in the following statement:

$$\neg(\forall s \exists \sigma (p \in L_s \Rightarrow p \in K_\sigma))$$

where s is the squareness factor of a logarithmic squircle, σ is the squareness factor of some other squircle, p is a point on a logarithmic squircle, L_s is a logarithmic squircle with squareness factor s, and K_σ is a squircle of some known type with squareness factor σ. Put another way, it must not be true that every logarithmic squircle is equivalent to some other type of squircle. Of course, there are theoretically infinitely many types of squircles, so this demonstration of non-equivalence will be limited in scope to the types of squircles that the author is aware of. Those types are Lamé squircles, Fernandez-Guasti squircles, oblique squircles [8], weighted squircles [9], and complex squircles [10].

5.1 Lamé, Fernandez-Guasti, and Oblique

To prove that logarithmic squircles are not equivalent to Lamé, Fernandez-Guasti, or oblique squircles, an image will be used to show that each of the three other types intersects the logarithmic squircle. If they intersect the logarithmic squircle, it will guarantee that no squareness factor could possibly make any of them equivalent to a logarithmic squircle. To set up the image, a purple logarithmic squircle with squareness factor 2, a green Lamé squircle with squareness factor 7, a black Fernandez-Guasti squircle with squareness factor 0.98, and an orange oblique squircle with squareness factor 2.83 are drawn. The first three are made such that they have squircular radii of 1. Before any of the other steps can be taken, the oblique squircle must be rotated and dilated so that it aligns with the other squircles:

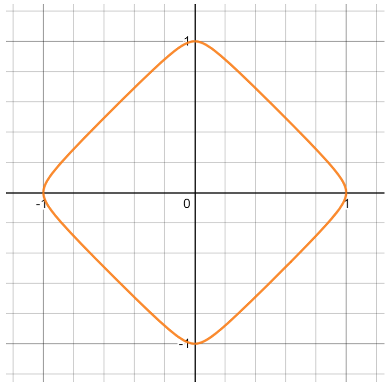

Figure 3: An oblique squircle [3]

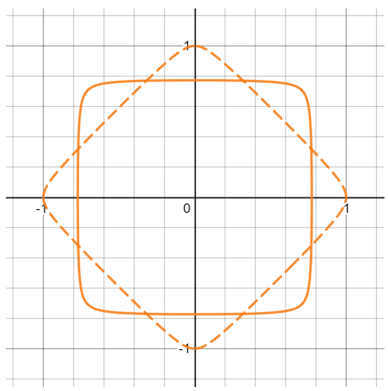

Figure 4: Rotating the oblique squircle by $\frac{\pi}{4}$ radians [3]

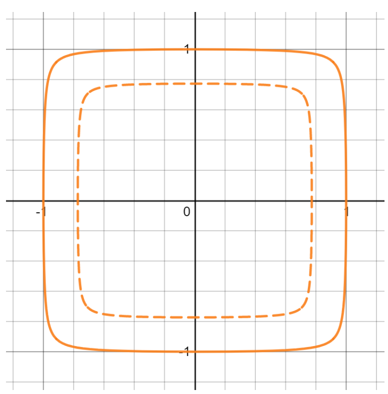

Figure 5: Dilating the oblique squircle such that its radius is 1 [3]

All four squircles can now be graphed at once for comparison:

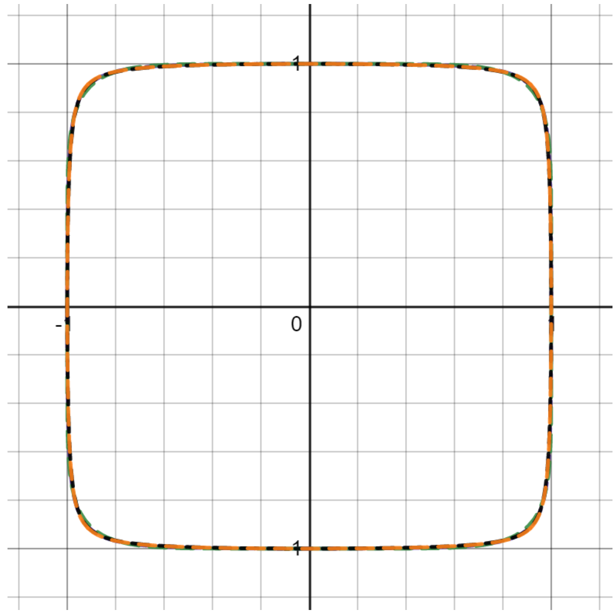

Figure 6: Four flavors of squircle on the same set of axes [3]

This graph must be enlarged to see the intersections:

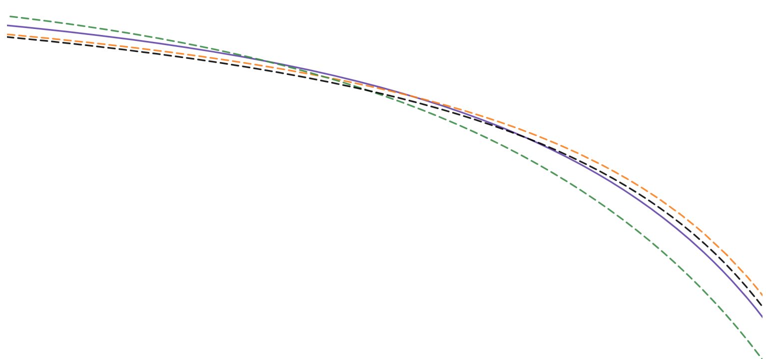

Figure 7: The other squircles (dashed) intersect the logarithmic squircle (solid) [3]

Assuming that one of the squircles is equivalent to a logarithmic squircle, increasing its squareness factor will not decrease its function, per the proof in 4.1.4. For the same reason, decreasing the squareness factor will not increase any values. However, the picture shows that some points on the squircle are below the logarithmic squircle and some are above it. If increasing the squareness factor cannot decrease any points, but decreasing it cannot increase any, then there is no way for all of the points on the "equivalent" squircle to line up with the logarithmic squircle. Hence, they are not equivalent by contradiction.

5.2 Weighted

Weighted squircles are visually different from the other types of squircles because of their pointed corners. Nevertheless, showing that a weighted squircle intersects the logarithmic squircle from above will be sufficient to show that they are non-equivalent. The reasoning is the same as that of the previous proof by contradiction. It should be briefly noted that these squircles were not called "weighted" squircles by the authors of [9], but they are called so in this paper because they are derived by taking a weighted average.

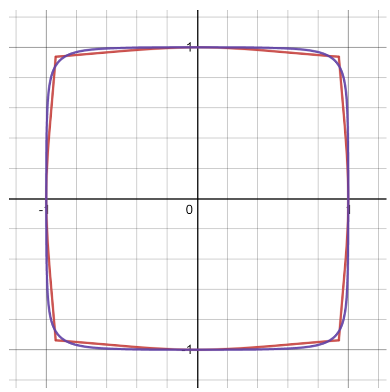

Figure 8: The logarithmic squircle intersects a weighted squircle [3]

5.3 Complex

The author does not have the technological capability of graphing a complex squircle, so a well-substantiated conjecture that these curves are not equivalent to logarithmic squircles will be offered instead. Note that a complex squircle is defined parametrically [10] as so:

$$x(t) = Re(\Omega(qe^{it}))\frac{r}{\Omega(q)}$$

$$y(t) = Im(\Omega(qe^{it}))\frac{r}{\Omega(q)}$$

where q is the squareness factor, r is the squircular radius, and $\Omega(qe^{it})$ is the following function of qe^{it}:

$$\Omega(w) = 1 - i - \frac{\sqrt{-2i}}{K_e}F(\cos^{-1}(w\sqrt{i}, \frac{1}{\sqrt{2}}))$$

In the above function, K_e is a constant, and F is a special function called the Legendre Elliptic Integral of the first kind. If this shape were equivalent to a logarithmic squircle with squareness factor s, it would imply that $q(x(t), s) = y(t)$ for some s and q. This would further imply that

$$q(x(t), s) = Im(\Omega(qe^{it}))\frac{r}{\Omega(q)}$$

The right-hand side of the equation involves the special function F, but the left-hand side can be written using only standard functions and operations. On this basis, the conjecture is made that the equation must be false, and thus logarithmic squircles and complex squircles must not be equivalent.

6 Conclusion

Several properties of logarithmic squircles have been found over the course of this investigation. It has been proven that they uphold the first, second, third, and fourth properties of general squircles, as stated in section 3.2. It has also been proven that logarithmic squircles have the property of non-equivalence to Lamé, Fernandez-Guasti, oblique, and weighted squircles, which shows that they are truly a novel class of curve. This property means that it is impossible to overlay a logarithmic squircle onto one of those squircles by changing the squareness factor. Non-equivalence to complex squircles was not proven, although it was conjectured. This is an area where the investigation could have been improved, but its proof may have been beyond the scope of this paper. Nevertheless, the theorems that were proven suffice to answer the question “What are the properties of logarithmic squircles?” Future areas of exploration may include finding the exact area or circumference of a logarithmic squircle. However, it appears that there may be no way to do so in terms of standard functions. It is possible to create a function that determines the area by definition, such as

$$J(s) = 2\int_{-1}^{1} q(x, s)\, dx$$

but an answer in terms of a special function like the error function may reveal more about the nature of logarithmic squircles. Another approach to expanding the investigation could be numerical approximation. The first few digits of the area of a logarithmic squircle with $r = 1$ and $s = 1$ are given by

$$J(1) \approx 3.47731$$

and it could be interesting to determine an infinite series that generates more of them. Yet again, the investigation could be expanded by examining practical applications of this class of squircle. One place they could be used would be graphic design, which is where other classes of squircles are frequently seen. For example, Apple uses a type of squircle for their app icons, so logarithmic squircles could presumably be used similarly. Logarithmic squircles and the function $q(x, s)$ could also potentially be used to stretch round images in more square-like forms. Additionally, the definition of a general squircle could serve as a template for future papers on squircles, which would help the mathematical community by making the term "squircle" more rigorous. Finally, it is possible that some of the seemingly minor work done in this paper, like the limits, derivatives, or inequalities, could serve a different purpose for a field unrelated to squircles.

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